

ECS 203 - Part 2A - For CPE2

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March 18, 2015

Part 1

Part 2

DC

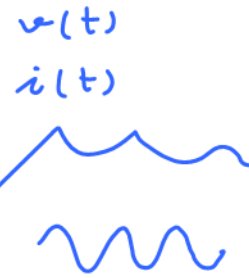
time-varying

constant voltage/current
(V) (I)

not varying with time

CHAPTER 6

arbitrary
sinusoidal



Energy Storage Elements: Capacitors and Inductors

To this point in our study of electronic circuits, time has not been important. The analysis and designs we have performed so far have been static, and all circuit responses at a given time have depended only on the circuit inputs at that time. In this chapter, we shall introduce two important passive circuit elements: the capacitor and the inductor.

6.1. Introduction and a **Mathematical Fact**

6.1.1. Capacitors and inductors, which are the electric and magnetic duals of each other, differ from resistors in several significant ways.

- Unlike resistors, which dissipate energy, capacitors and inductors do not dissipate but store energy, which can be retrieved at a later time. They are called **storage elements**.
- Furthermore, their branch variables do not depend algebraically upon each other. Rather, their relations involve temporal derivatives and integrals. Thus, the analysis of circuits containing capacitors and inductors involve **differential equations** in time.

Resistor:

$$V = IR$$

$$v(t) = i(t)R$$

Capacitor: $i(t) = C \frac{d}{dt} v(t)$

6.1.2. An important mathematical fact: Given

Two useful forms of the solution:

Use this when the initial condition $f(t_0) = \dots$ is given/known

$$\frac{d}{dt} f(t) = g(t),$$

$$f(t) = \int_{-\infty}^t g(\tau) d\tau$$

$$= \int_{t_0}^t g(\tau) d\tau + f(t_0)$$

Use this when you know the complete history ($t_0 \rightarrow -\infty$) of $g(t)$.

$$= \int_{-\infty}^{t_0} g(\tau) d\tau$$

initial condition (IC)

6.2. Capacitors

6.2.1. A capacitor is a passive element designed to store energy in its electric field. The word capacitor is derived from this element's capacity to store energy.

6.2.2. When a voltage source $v(t)$ is connected across the capacitor, the amount of charge stored, represented by q , is directly proportional to $v(t)$, i.e.,

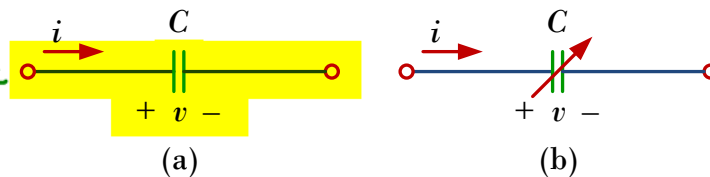
$$q(t) = Cv(t)$$

where C , the constant of proportionality, is known as the **capacitance** of the capacitor.

- The unit of capacitance is the **farad (F)** in honor of Michael Faraday.
- 1 farad = 1 coulomb/volt.

6.2.3. Circuit symbol for capacitor of C farads:

Note that we still assume passive sign convention.



6.2.4. Since $i = \frac{dq}{dt}$, then the current-voltage relationship of the capacitor is

$$(6.2) \quad i = C \frac{dv}{dt}$$

Note that in (6.2), the capacitance value C is constant (time-invariant) and that the current i and voltage v are both functions of time (time-varying). So, in fact, the full form of (6.2) is

$$i(t) = C \frac{d}{dt} v(t) = \frac{d}{dt} (C v(t))$$

Hence, the voltage-current relation is

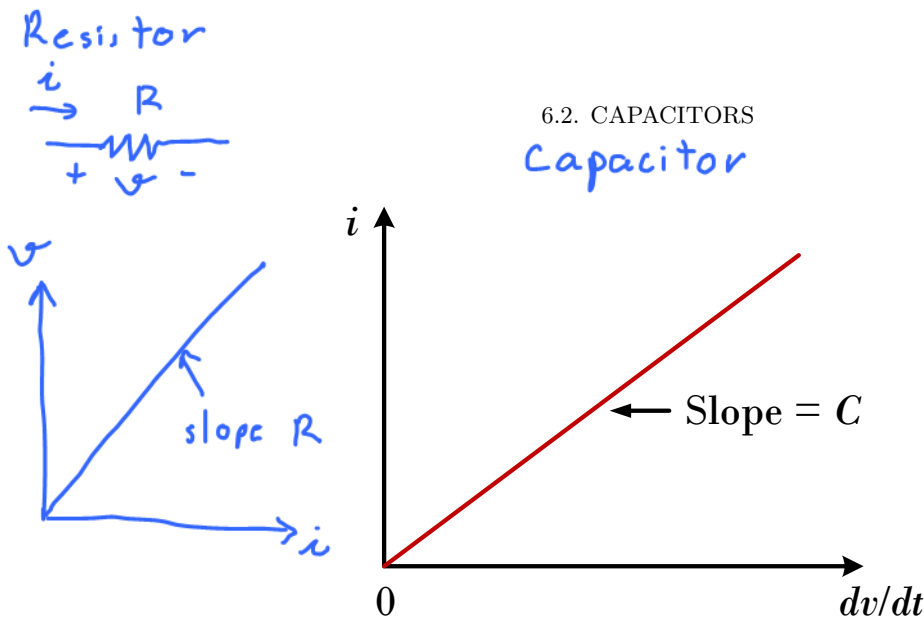
$$v(t) = \frac{1}{C} \int_{t_0}^t i(\tau) d\tau + v(t_0)$$

Recall $\frac{d}{dt} v(t) = \frac{1}{C} i(t)$

$\frac{d}{dt} f(t) = g(t)$

$\Rightarrow f(t) = \int_{t_0}^t g(\tau) d\tau + f(t_0)$

$v(t) = \int_{t_0}^t \frac{1}{C} i(\tau) d\tau + v(t_0)$



where $v(t_0)$ is the voltage across the capacitor at time t_0 . Note that capacitor voltage depends on the past history of the capacitor current. Hence, the capacitor has *memory*.

6.2.5. The **instantaneous power** delivered to the capacitor is

$$p(t) = i(t) \times v(t) = \left(C \frac{d}{dt} v(t) \right) v(t).$$

The **energy stored in the capacitor** is

$$w(t) = \int_{-\infty}^t p(\tau) d\tau = \frac{1}{2} C v^2(t).$$

In the above calculation, we assume $v(-\infty) = 0$, because the capacitor was uncharged at $t = -\infty$.

6.2.6. Typical values

- Capacitors are commercially available in different values and types.
- Typically, capacitors have values in the picofarad (pF) to microfarad (μF) range.
- For comparison, two pieces of insulated wire about an inch long, when twisted together, will have a capacitance of about 1 pF.

6.2.7. Two important implications of (6.2):

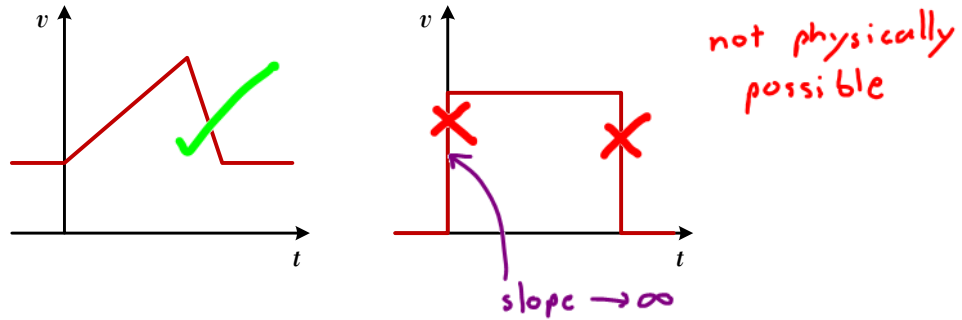
- A capacitor is an open circuit to dc.

When the voltage across a capacitor is not changing with time (i.e., dc voltage), its derivative wrt. time is $\frac{dv}{dt} = 0$ and hence the current through the capacitor is $i(t) = C \frac{dv}{dt} = C \times 0 = 0$.

$$i = C \frac{dv}{dt} = 0 \equiv \text{open circuit}$$

(b) The voltage across a capacitor cannot jump (change abruptly)

Because $i = C \frac{dv}{dt}$, a discontinuous change in voltage requires an infinite current, which is physically impossible.



6.2.8. Remark: An ideal capacitor does not dissipate energy. It takes power from the circuit when storing energy in its field and returns previously stored energy when delivering power to the circuit.

EXAMPLE 6.2.9. If a $10 \mu F$ is connected to a voltage source with

$$v(t) = 50 \sin(2000t) \text{ V}$$

determine the current through the capacitor.

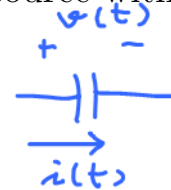
$$\frac{d}{dt} \sin t = \cos t$$

$$i = C \frac{dv}{dt}$$

$$\frac{d}{dt} \cos t = -\sin t$$

$$= 10 \times 10^{-6} \times 50 \cos(2000t) \times 2000$$

$$= \cos(2000t) \text{ A}$$



EXAMPLE 6.2.10. Determine the voltage across a $2\text{-}\mu F$ capacitor if the current through it is

$$i(t) = 6e^{-3000t} \text{ mA}$$

Assume that the initial capacitor voltage (at time $t = 0$) is zero.

$$\frac{d}{dt} e^t = e^t$$

$$\frac{d}{dt} e^{at} = a e^{at}$$

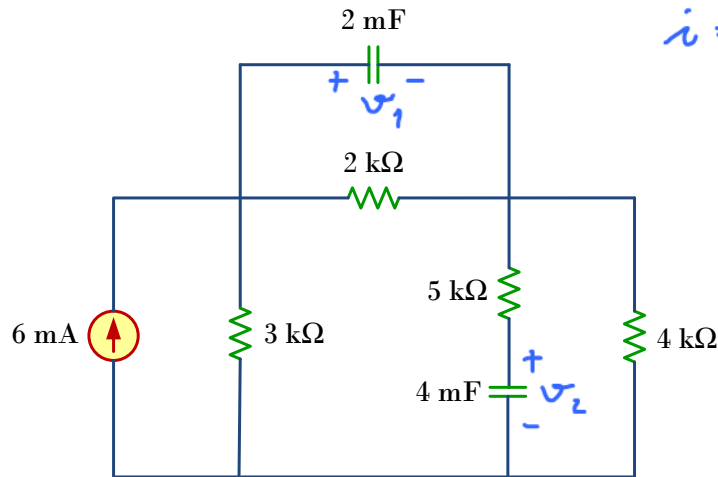
$$\int e^{at} dt = \frac{1}{a} e^{at}$$

$$v(t) = \frac{1}{C} \int_{t_0}^t i(\tau) d\tau + v(t_0)$$

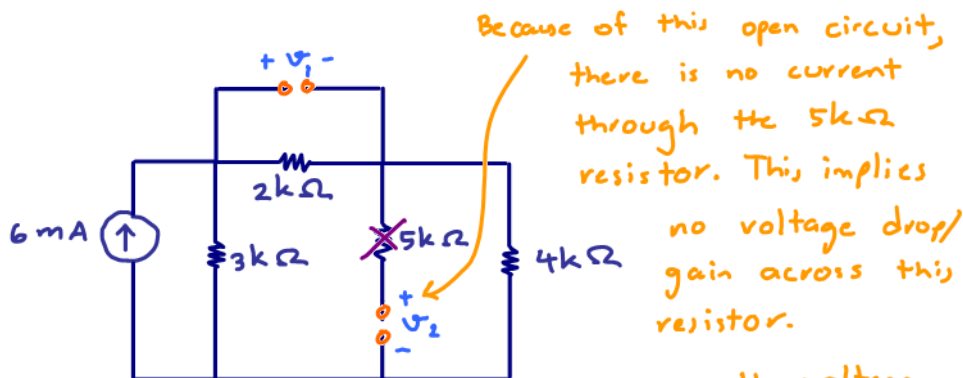
$$= \frac{1}{C} \int_0^t i(\tau) d\tau + 0 = \frac{1}{2 \times 10^{-6}} \int_0^t 6 e^{-3000\tau} \times 10^{-3} d\tau$$

$$= \frac{3}{10^{-3}} \times \frac{1}{-3000} e^{-3000\tau} \Big|_0^t = -e^{-3000t} - (-1) = 1 - e^{-3000t} \text{ V}$$

EXAMPLE 6.2.11. Obtain the energy stored in each capacitor in the figure below under dc conditions.

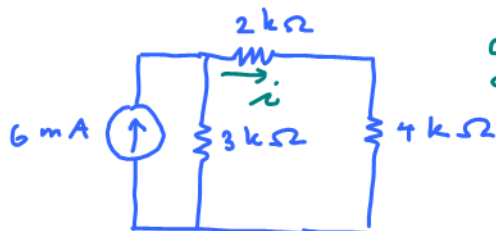


$i = C \frac{dv}{dt} \rightarrow 0$
 $= 0 \Rightarrow$ open circuit
 For dc,
 capacitor \rightarrow open circuit



Because of this open circuit, there is no current through the $5k\Omega$ resistor. This implies no voltage drop/gain across this resistor.

$v_2 =$ the voltage across the $4k\Omega$ resistor.



$$i = \frac{3k}{3k + (2k + 4k)} \times 6mA$$

$$= 2mA$$

$$v_1 = i \times 2k = 2mA \times 2k = 4V$$

$$v_2 = i \times 4k = 2mA \times 4k = 8V$$

$$w_1 = \frac{1}{2} C_1 v_1^2 = \frac{1}{2} \times 2 \times 10^{-3} \times 4^2 = 16mJ$$

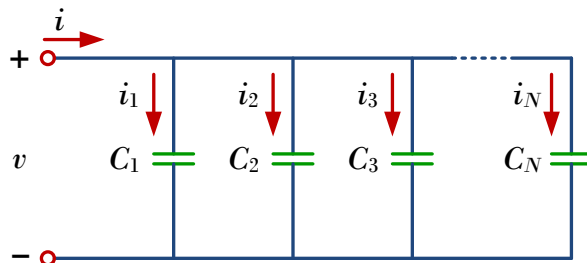
$$w_2 = \frac{1}{2} C_2 v_2^2 = \frac{1}{2} \times 4 \times 10^{-3} \times 8^2 = 128mJ$$

6.3. Series and Parallel Capacitors

We know from resistive circuits that series-parallel combination is a powerful tool for simplifying circuits. This technique can be extended to series-parallel connections of capacitors, which are sometimes encountered. We desire to replace these capacitors by a single equivalent capacitor C_{eq} .

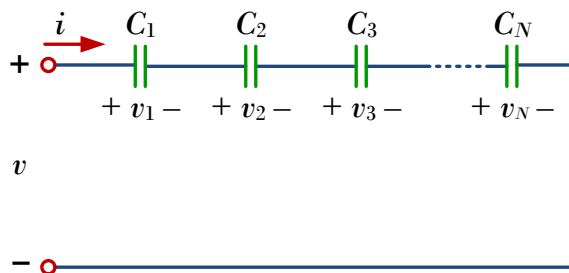
6.3.1. The equivalent capacitance of N parallel-connected capacitors is the sum of the individual capacitance.

$$C_{eq} = C_1 + C_2 + \cdots + C_N$$

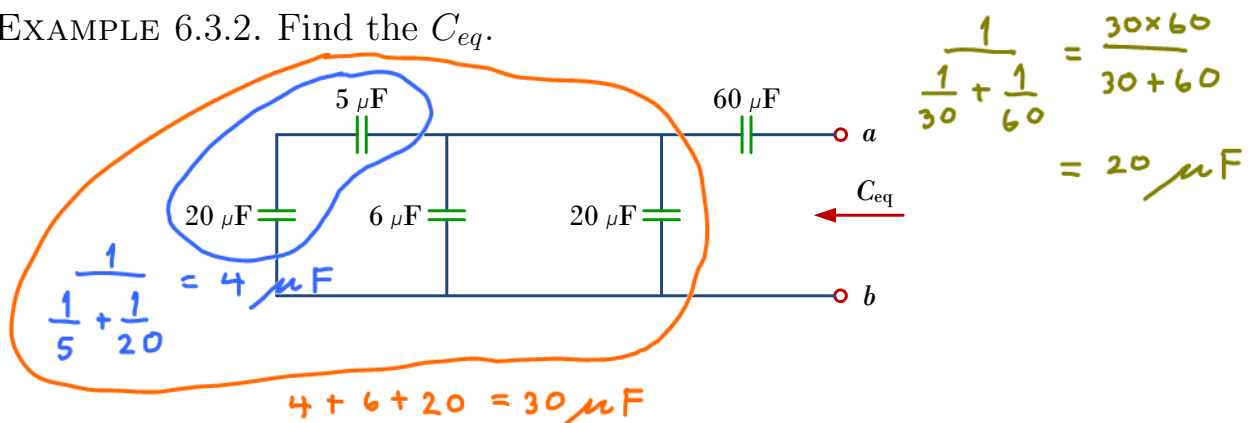


The equivalent capacitance of N series-connected capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances.

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}$$



EXAMPLE 6.3.2. Find the C_{eq} .

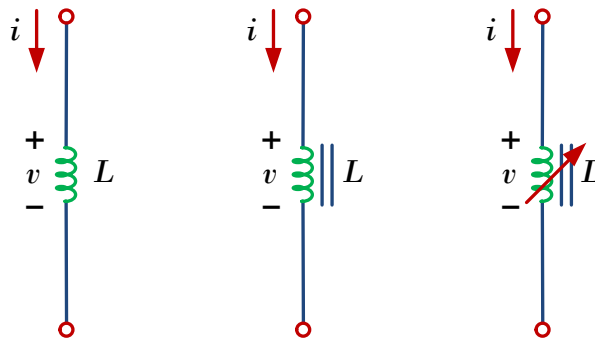


6.4. Inductors

6.4.1. An inductor is a passive element designed to store energy in its magnetic field.

6.4.2. Inductors find numerous applications in electronic and power systems. They are used in power supplies, transformers, radios, TVs, radars, and electric motors.

6.4.3. Circuit symbol of inductor:



6.4.4. If a current is allowed to pass through an inductor, the voltage across the inductor is directly proportional to the time rate of change of the current, i.e.,

$$(6.3) \quad v(t) = L \frac{d}{dt} i(t),$$

where L is the constant of proportionality called the **inductance** of the inductor. The unit of inductance is **henry** (H), named in honor of Joseph Henry.

- 1 henry equals 1 volt-second per ampere.

$$\frac{di}{dt} = \frac{1}{L} v$$

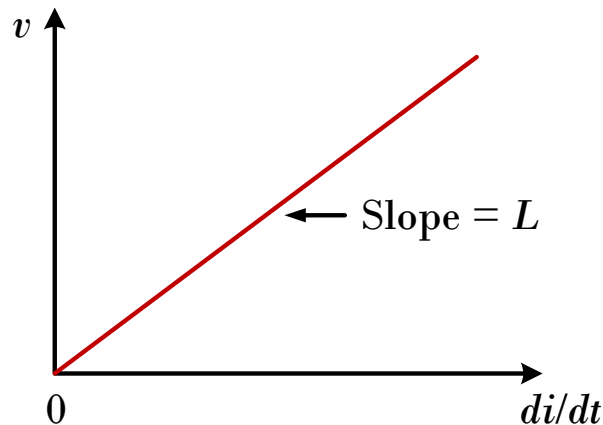
6.4.5. By integration, the current-voltage relation is

$$i(t) = \frac{1}{L} \int_{t_o}^t v(\tau) d\tau + i(t_o),$$

where $i(t_o)$ is the current at time t_o .

6.4.6. The instantaneous **power** delivered to the inductor is

$$p(t) = v(t) \times i(t) = \left(L \frac{d}{dt} i(t) \right) i(t)$$



The energy stored in the inductor is

$$w(t) = \int_{-\infty}^t p(\tau) d\tau = \frac{1}{2} Li^2(t).$$

6.4.7. Like capacitors, commercially available inductors come in different values and types. Typical practical inductors have inductance values ranging from a few microhenrys (μH), as in communication systems, to tens of henrys (H) as in power systems.

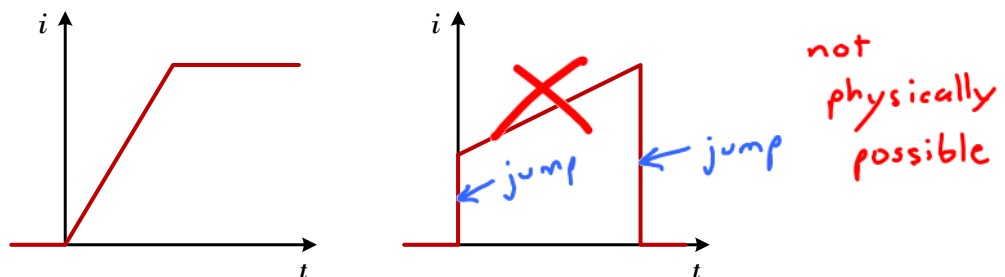
6.4.8. Two important implications of (6.3):

- (a) An inductor acts like a short circuit to dc.

When the current through an inductor is not changing with time (i.e., dc current), its derivative wrt. time is $\frac{di}{dt} = 0$ and hence the voltage across the inductor is $v(t) = L \frac{di}{dt} = L \times 0 = 0$.

- (b) The current through an inductor cannot change instantaneously.

This opposition to the change in current is an important property of the inductor. A discontinuous change in the current through an inductor requires an infinite voltage, which is not physically possible.



6.4.9. Remark: The ideal inductor does not dissipate energy. The energy stored in it can be retrieved at a later time. The inductor takes

power from the circuit when storing energy and delivers power to the circuit when returning previously stored energy.

EXAMPLE 6.4.10. If the current through a 1-mH inductor is $i(t) = 20 \cos(100t)$ mA, find the terminal voltage and the energy stored.

Terminal voltage of an element is simply the voltage across (the two terminals) of that element.

$$\begin{aligned} v(t) &= L \frac{di(t)}{dt} = 10^{-3} \times \frac{d}{dt} 20 \cos(100t) \times 10^{-3} \\ &= 10^{-6} \times 20 \times (-1) \times \sin(100t) \times \underbrace{100}_{\text{chain rule}} \\ &= -2 \sin(100t) \text{ mV} \\ w(t) &= \frac{1}{2} L i^2(t) = \frac{1}{2} \times 10^{-3} \times (20 \cos(100t) \times 10^{-3})^2 \\ &= \frac{1}{2} \times 10^{-3} \times \frac{200}{2} \times \cos^2(100t) \times 10^{-6} = 0.2 \cos^2(100t) \mu\text{J} \end{aligned}$$

EXAMPLE 6.4.11. Find the current through a 5-H inductor if the voltage across it is

$$v(t) = \begin{cases} 30t^2, & t > 0 \\ 0, & t < 0 \end{cases}$$

$$w(t) = \frac{1}{2} L i^2$$

In addition, find the energy stored within $0 < t < 5$ s.

$$v = L \frac{di}{dt} \Rightarrow \frac{di}{dt} = \frac{1}{L} v \Rightarrow i(t) = \frac{1}{L} \int_{t_0}^t v(\tau) d\tau + i(t_0)$$

$$\Rightarrow w(5) - w(0)$$

$$i(t) = \frac{1}{5} \times \int_{-\infty}^t v(\tau) d\tau = \begin{cases} 0, & t < 0 \\ \frac{1}{5} \left(\int_{-\infty}^0 v(\tau) d\tau + \int_0^t v(\tau) d\tau \right), & t \geq 0 \end{cases}$$

For $t \geq 0$,

$$i(t) = \frac{1}{5} \int_0^t 30\tau^2 d\tau = \frac{1}{5} \times 30 \left. \frac{\tau^3}{3} \right|_0^t = 2t^3 \text{ A}$$

$$i(0) = 0 \text{ A}$$

$$w(0) = \frac{1}{2} \times 5 \times 0^2 = 0 \text{ J}$$

$$i(5) = 2 \times 5^3$$

$$w(5) = \frac{1}{2} \times 5 \times 2^2 \times 5^6 = 2 \times 5^7 \text{ J}$$

$$\left. \begin{array}{l} w(5) - w(0) \\ 2 \times 5^7 \text{ J} \end{array} \right\}$$

EXAMPLE 6.4.12. The terminal voltage of a 2-H inductor is $v(t) = 10(1 - t)$ V. Find the current flowing through it at $t = 4$ s and the energy stored in it within $0 < t < 4$ s. Assume $i(0) = 2$ A. $t_0 = 0$

$$v = L \frac{di}{dt} \Rightarrow i(t) = \frac{1}{L} \int_{t_0}^t v(\tau) d\tau + i(t_0)$$

$$= \frac{1}{2} \int_0^4 10(1 - \tau) d\tau + 2 = 5 \left(\tau - \frac{\tau^2}{2} \right) \Big|_0^4 + 2$$

$$= 5 \left(4 - \frac{4^2}{2} \right) + 2$$

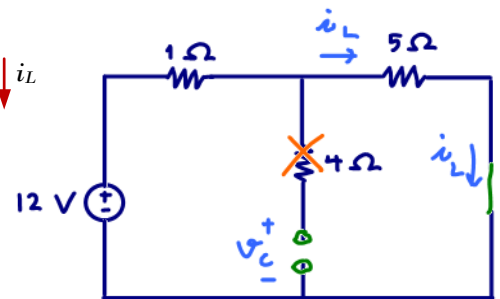
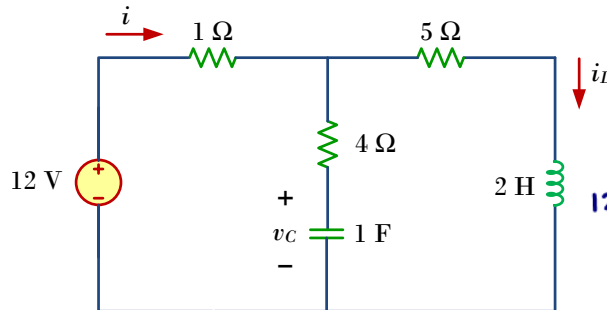
$$i(4) = 5 \left(4 - \frac{4^2}{2} \right) + 2 = -18 \text{ A}$$

$$w(4) - w(0)$$

$$= \frac{1}{2} \times 2 \times (-18)^2$$

$$- \frac{1}{2} \times 2 \times 2^2 = 320 \text{ J}$$

EXAMPLE 6.4.13. Determine v_C , i_L and the energy stored in the capacitor and inductor in the following circuit under dc conditions.



$v_C =$ voltage across the 5 Ω resistor

$$i_L = \frac{12}{1+5} = \frac{12}{6} = 2 \text{ A}$$

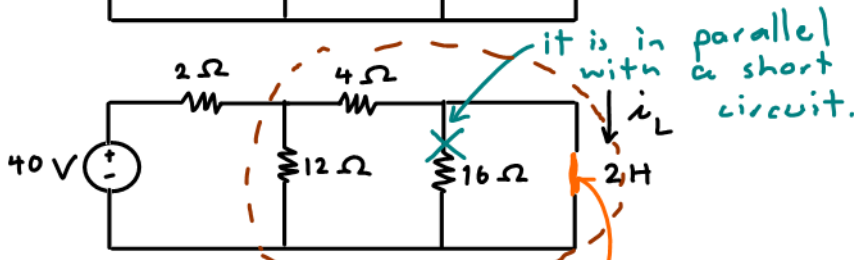
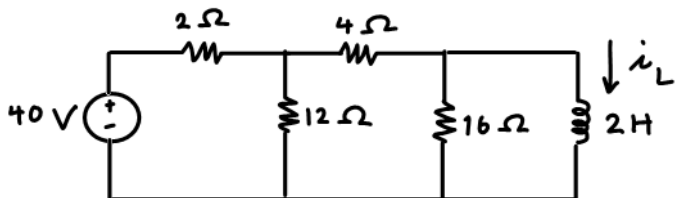
$$v_C = i_L \times 5 = 2 \times 5 = 10 \text{ V}$$

$$w_C = \frac{1}{2} C v_C^2 = \frac{1}{2} \times 1 \times 10^2 = 50 \text{ J}$$

$$w_L = \frac{1}{2} L i_L^2 = \frac{1}{2} \times 2 \times 2^2 = 4 \text{ J}$$

Extra Ex.

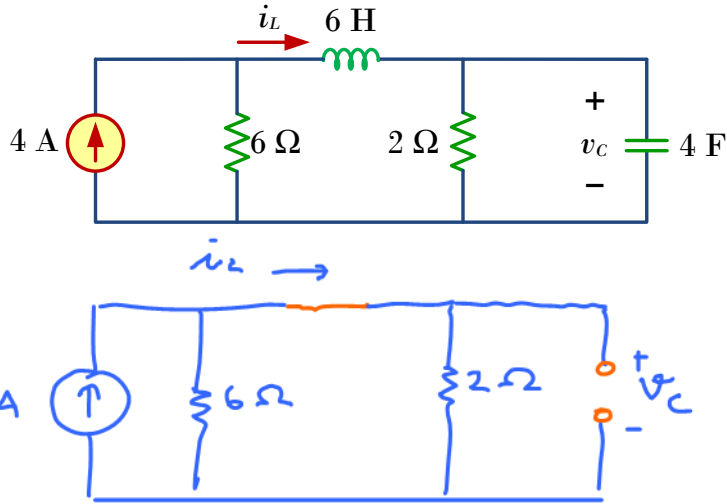
Find i_L under dc conditions



$12 \parallel 4 = 3 \Omega$ under dc conditions

voltage divider $V_{3\Omega} = \frac{3}{2+3} \times 40$ inductor \rightarrow short circuit
 $i_L = \frac{V_{4\Omega}}{4\Omega} = \frac{24}{4} = 6 \text{ A}$

EXAMPLE 6.4.14. Determine v_C , i_L and the energy stored in the capacitor and inductor in the following circuit under **dc conditions**.



$$i_L = i_{2\Omega} = \frac{6}{6+2} \times 4 = 3 \text{ A} \Rightarrow w_L = \frac{1}{2} L i_L^2 = \frac{1}{2} \times 6 \times 3^2 = 27 \text{ J}$$

current divider

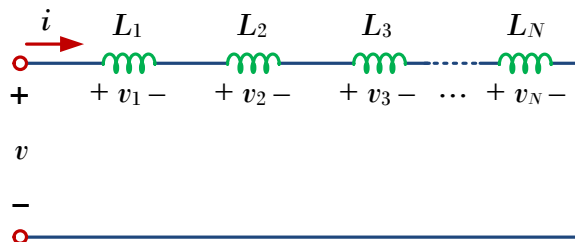
$$v_C = v_{2\Omega} = i_L \times 2 = 3 \times 2 = 6 \text{ V} \Rightarrow w_C = \frac{1}{2} C v_C^2 = \frac{1}{2} \times 4 \times 6^2 = 72 \text{ J}$$

Ohm's law

6.5. Series and Parallel Inductors

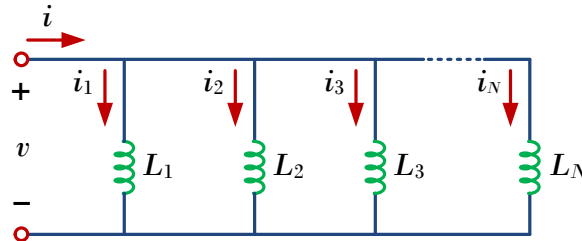
6.5.1. The equivalent inductance of N series-connected inductors is the sum of the individual inductances, i.e.,

$$L_{eq} = L_1 + L_2 + \dots + L_N$$



6.5.2. The equivalent inductance of N parallel inductors is the reciprocal of the sum of the reciprocals of the individual inductances, i.e.,

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}$$



6.5.3. Remark: Note that

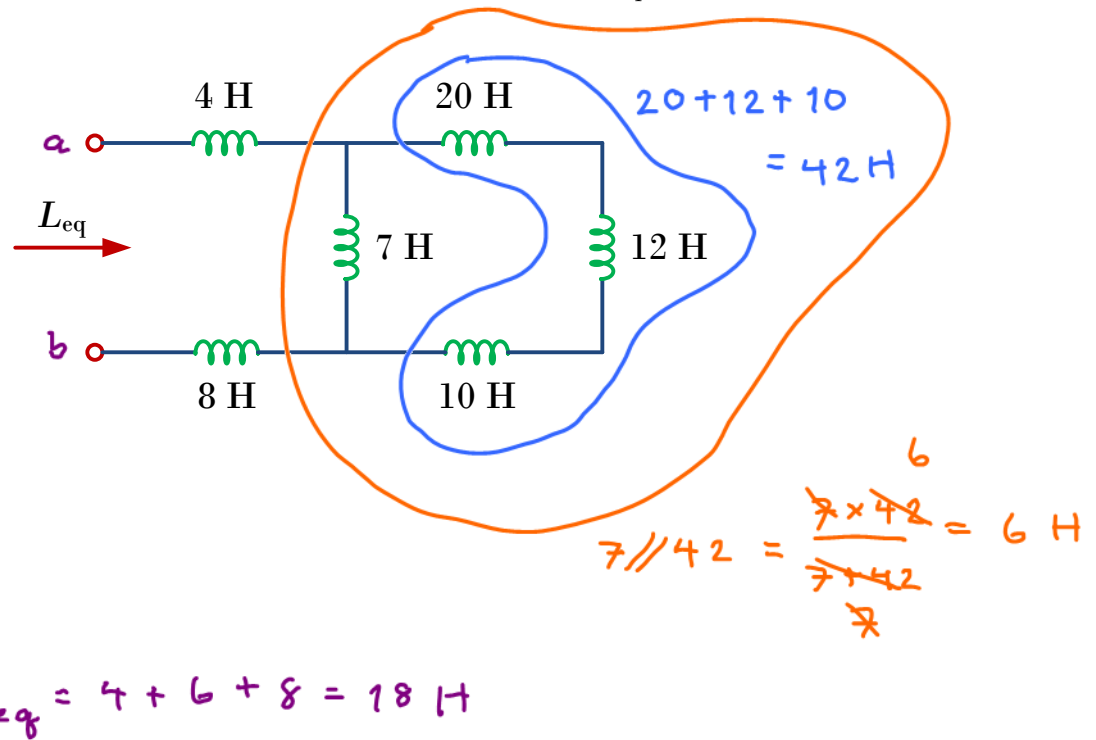
- inductors in series are combined in exactly the same way as resistors in series and
- inductors in parallel are combined in the same way as resistors in parallel.

Important characteristics of the basic elements.[†]

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v - i :	$v = iR$	$v = \frac{1}{C} \int_{t_0}^t i dt + v(t_0)$	$v = L \frac{di}{dt}$
i - v :	$i = v/R$	$i = C \frac{dv}{dt}$	$i = \frac{1}{L} \int_{t_0}^t v dt + i(t_0)$
p or w :	$p = i^2 R = \frac{v^2}{R}$	$w = \frac{1}{2} C v^2$	$w = \frac{1}{2} L i^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1 L_2}{L_1 + L_2}$
At dc: under dc conditions Circuit variable that cannot change abruptly:	Same	Open circuit	Short circuit
	Not applicable	v	i

[†] Passive sign convention is assumed.

EXAMPLE 6.5.4. Find the equivalent inductance L_{eq} of the circuit shown below.



6.6. Applications: Integrators and Differentiators

6.6.1. Capacitors and inductors possess the following three special properties that make them very useful in electric circuits:

- The capacity to store energy makes them useful as temporary voltage or current sources. Thus, they can be used for generating a large amount of current or voltage for a short period of time.
- Capacitors oppose any abrupt change in voltage, while inductors oppose any abrupt change in current. This property makes inductors useful for spark or arc suppression and for converting pulsating dc voltage into relatively smooth dc voltage.
- Capacitors and inductors are frequency sensitive. This property makes them useful for frequency discrimination.

The first two properties are put to use in dc circuits, while the third one is taken advantage of in ac circuits.

In this final part of the chapter, we will consider two applications involving capacitors and op amps: integrator and differentiator.

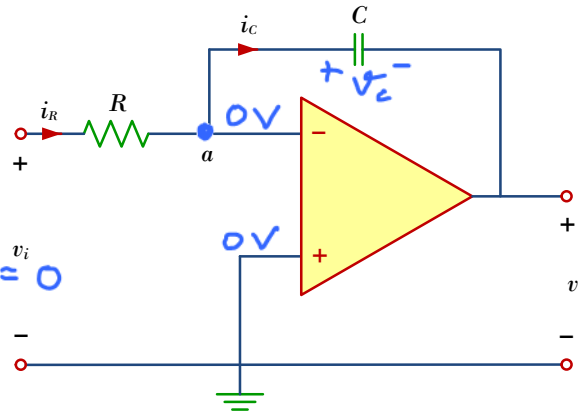
inverting

6.6.2. An **integrator** is an op amp circuit whose output is proportional to the integral of the input signal. We obtain an integrator by replacing the feedback resistor R_f in the inverting amplifier by a capacitor.

KCL @ "-"

$$\frac{0 - v_i}{R} + i_c = 0$$

$$-\frac{v_i}{R} + \left(-C \frac{dv_o}{dt}\right) = 0$$



$$i_c = C \frac{dv_c}{dt} = C \frac{d(0 - v_o)}{dt}$$

This gives

$$\frac{d}{dt}v_o(t) = -\frac{1}{RC}v_i(t),$$

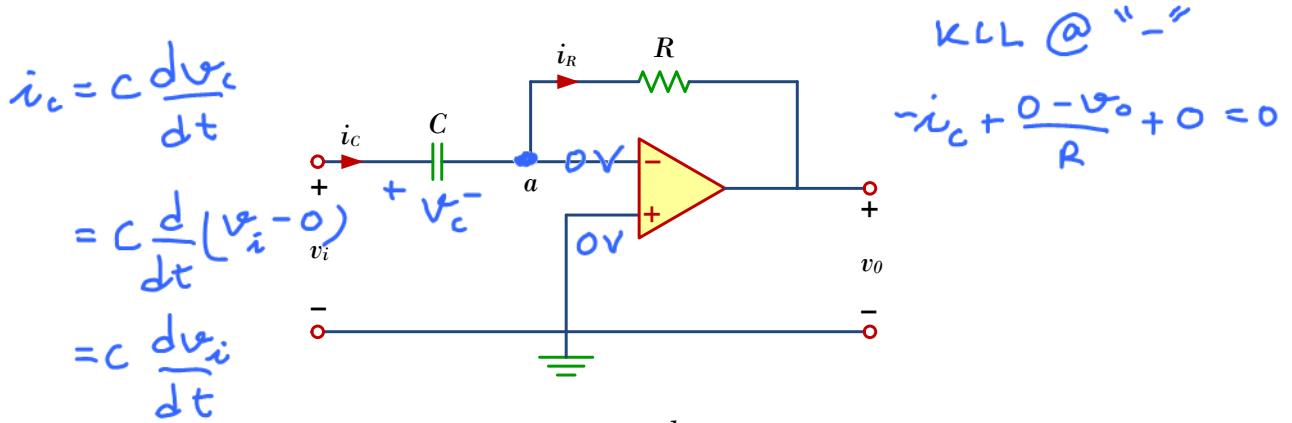
which implies

$$v_o(t) = -\frac{1}{RC} \int_{t_0}^t v_i(\tau) d\tau + v_o(t_0)$$

Formula given @ the beginning of this chapter.

- To ensure that $v_o(0) = 0$, it is always necessary to discharge the integrator's capacitor prior to the application of a signal.
- In practice, the op amp integrator requires a feedback resistor to reduce dc gain and prevent saturation. Care must be taken that the op amp operates within the linear range so that it does not saturate.

6.6.3. A **differentiator** is an op amp circuit whose output is proportional to the differentiation of the input signal. We obtain a differentiator by replacing the input resistor in the inverting amplifier by a capacitor. This gives



$$v_o(t) = -RC \frac{d}{dt} v_i(t).$$

- Differentiator circuits are electronically unstable because any electrical noise within the circuit is exaggerated by the differentiator. For this reason, the differentiator circuit above is not as useful and popular as the integrator. It is seldom used in practice.